Chapter 10 Systems of Equations and Inequalities

10.1

5. $\begin{cases} x - y = 1 \\ 4x + 3y = 18 \end{cases}$

$x = y + 1$

$4(y+1) + 3y = 18$

$4y + 4 + 3y = 18$

$7y = 14$

$y = 2$

$x = (2) + 1$

$= 3$

$\therefore x = 3, y = 2$

9. $\begin{cases} 3x + 4y = 10 \\ x - 4y = -2 \end{cases}$

Eq1 + Eq2 : $4x = 8$

$x = 2$

$2 - 4y = -2$ $\therefore (2, 1)$

$-4y = -4$

$y = 1$

10.3 Partial Fractions

1. Distinct Linear Factors
2. Repeated Linear Factors
3. Irreducible Quadratic Factors
4. Repeated Irreducible Quadratic Factors

1. $r(x) = \dfrac{4}{x(x-2)^2}$

$$\frac{4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$4 = A(x-2)^2 + Bx(x-2) + Cx$$

$$4 = Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx$$

$$4 = (A+B)x^2 + (-4A - 2B + C)x + 4A$$

$A = 1,$

$A + B = 0$

$B = -1,$

$-4A - 2B + C = 0$

$-4 + 2 + C = 0$

$C = 2$

$$r(x) = \frac{1}{x} - \frac{1}{x-2} + \frac{2}{(x-2)^2}$$

$\boxed{2}$ (iii)

$$r(x) = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

24/12/2023

2. $r(x) = \dfrac{2x+8}{(x-1)(x^2+4)}$

(ii) $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

3. $\dfrac{1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

4. $\dfrac{x}{x^2+3x-4} = \dfrac{x}{(x+4)(x-1)}$

$$\frac{x}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

5. $\dfrac{x^2-3x+5}{(x-2)^2(x+4)} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+4}$

13. $$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2 = A(x+1) + B(x-1)$$ 25/12/23

$$2 = Ax + A + Bx - B$$

$$2 = (A+B)x + A - B$$

$$A + B = 0 \quad — ①$$

$$A - B = 2 \quad — ②$$

① + ②: $2A = 2$

$$A = 1$$

$$1 + B = 0$$

$$B = -1$$

$$\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

14. $$\frac{2x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2x = A(x+1) + B(x-1)$$

$$2x = (A+B)x + A - B$$

$$A + B = 2 \quad — ①$$

$$A - B = 0 \quad — ②$$

① + ② :

$$2A = 2$$

$$A = 1$$

$$1 - B = 0$$

$$B = 1$$

$$\therefore \frac{2x}{(x-1)(x+1)} = \frac{1}{x-1} + \frac{1}{x+1}$$

26/12/23

15. $$\frac{5}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

$$5 = Ax + 4A + Bx - B$$
$$= (A+B)x + 4A - B$$

$A + B = 0$

$4A - B = 5$

$$\frac{5}{(x-1)(x+4)} = \frac{1}{x-1} - \frac{1}{x+4}$$

$A + B = 0$

$B = -A$

$4A + A = 5$

$5A = 5$

$A = 1$

$B = -1$

16. $$\frac{x+6}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$$

$$x+6 = Ax + 3A + Bx$$

$$= (A+B)x + 3A$$

$$A+B = 1$$

$$3A = 6$$

$$A = 2$$

$$2+B = 1$$

$$B = -1$$

$$\frac{x+6}{x(x+3)} = \frac{2}{x} - \frac{1}{x+3}$$

17. $\dfrac{12}{x^2-9} = \dfrac{12}{(x+3)(x-3)}$

$$\frac{12}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

$$12 = A(x-3) + B(x+3)$$

$$12 = Ax - 3A + Bx + 3B$$

$$= (A+B)x + 3B - 3A$$

$A + B = 0$

$3B - 3A = 12$

$$\frac{12}{(x+3)(x-3)} = \frac{2}{x-3} - \frac{2}{x+3}$$

$A = -B$

$3B - 3(-B) = 12$

$6B = 12$

$B = 2$

$A = -2$

27/12/23

18. $$\frac{x-12}{x^2-4x} = \frac{x-12}{x(x-4)}$$

$$\frac{x-12}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$

$$x-12 = Ax - 4A + Bx$$
$$x-12 = (A+B)x - 4A$$

$$A+B = 1$$
$$12 = 4A$$
$$A = 3$$

$$\frac{x-12}{x(x-4)} = \frac{3}{x} - \frac{2}{x-4}$$

$$3+B = 1$$
$$B = -2$$

Chapter 11 Matrices and Determinants

1. Matrices and Systems of Linear Equations
2. The Algebra of Matrices
3. Inverses of Matrices and Matrix Equations
4. Determinants and Cramer's Rule

11.1 Matrices and Systems of Linear Equations

5. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$ 3 by 2

11. $\begin{aligned} 3x + y - z &= 2 \\ 2x - y \quad &= 1 \\ x \quad - z &= 3 \end{aligned}$

$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

(a) yes

(b) yes

(c) $x = -3$
$y = 5$

21. $\begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 1 & 1 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$

$$\downarrow \; 3R_1 + R_2 \to R_2$$

$$\begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 4 & 7 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

25. $$\begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) $$x - 2y + 4z = 3$$
$$y + 2z = 7$$
$$z = 2$$

(b) $y = 3$

$$x - 6 + 8 = 3$$
$$x = 1$$

$(1, 3, 2)$

29. $x - 2y + z = 1$

$y + 2z = 5$

$x + y + 3z = 8$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 8 \end{bmatrix}$$

$$\Big\downarrow R_3 - R_1 \to R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 2 & 7 \end{bmatrix}$$

$$\Big\downarrow R_3 - 3R_2 \to R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -4 & -8 \end{bmatrix}$$

$$\Big\downarrow -\frac{1}{4}R_3 \to R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$x - 2y + z = 1$

$y + 2z = 5$

$z = 2$

$y + 2(2) = 5$

$y = 1$

$x - 2(1) + 2 = 1$

$\therefore x = 1$

$\therefore (1, 1, 2)$

33. $x+2y-z=-2$

$x \quad +z=0$

$2x-y-z=-3$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & -3 \end{array}\right]$$

$$\Big\downarrow \ \frac{R_2-R_1 \to R_2}{R_3-2R_1 \to R_3}$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & -2 & 2 & 2 \\ 0 & -5 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow \ -\frac{1}{2}R_2$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -5 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow \ R_3+5R_2 \to R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

$$\Big\downarrow \ -\frac{1}{4}R_3 \to R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow \ \frac{R_2+R_3 \to R_2}{R_1+R_3 \to R_1}$$

$$\left[\begin{array}{cccc} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow \ R_1-2R_2 \to R_1$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ $x=-1$

$y=0$

$z=1$

39.
$$\begin{aligned} x+y+z&=2\\ y-3z&=1\\ 2x+y+5z&=0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{bmatrix}$$

$$\Big\downarrow\ R_3-2R_1 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -4 \end{bmatrix}$$

$$\Big\downarrow\ R_3+R_2 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$\therefore$ Inconsistent Linear System

40. $x \quad + 3z = 3$

$2x + y - 2z = 5$

$-y + 8z = 8$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 2 & 1 & -2 & 5 \\ 0 & -1 & 8 & 8 \end{bmatrix}$$

$$\Big\downarrow \; 2R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & -1 & 8 & 1 \\ 0 & -1 & 8 & 8 \end{bmatrix}$$

$$\Big\downarrow \; R_2 - R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & -1 & 8 & 1 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

$\therefore$ Inconsistent Linear System

41.

$$2x - 3y - 9z = -5$$
$$x + 3z = 2$$
$$-3x + y - 4z = -3$$

$$\begin{bmatrix} 2 & -3 & -9 & -5 \\ 1 & 0 & 3 & 2 \\ -3 & 1 & -4 & -3 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_1]{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 2 & -3 & -9 & -5 \\ -3 & 1 & -4 & -3 \\ 1 & 0 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow[R_1 - 2R_3 \to R_3]{3R_1 + 2R_2 \to R_2}$$

$$\begin{bmatrix} 2 & -3 & -9 & -5 \\ 0 & -7 & -35 & -21 \\ 0 & -3 & -15 & -9 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_1]{R_2 - \frac{7}{3}R_3 \to R_3}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{9}{2} & -\frac{5}{2} \\ 0 & -7 & -35 & -21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Dependent

$$\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{9}{2} & -\frac{5}{2} \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 + \frac{3}{2}R_2 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x + 3z = 2$$
$$y + 5z = 3$$

Let z be t,

$$x = 2 - 3t$$
$$y = 3 - 5t$$
$$z = t$$

42. $\begin{cases} x-2y+5z=3 \\ -2x+6y-11z=1 \\ 3x-16y+20z=-26 \end{cases}$

$$\begin{bmatrix} 1 & -2 & 5 & 3 \\ -2 & 6 & -11 & 1 \\ 3 & -16 & 20 & -26 \end{bmatrix} \xrightarrow[3R_1-R_3\to R_3]{2R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 2 & -1 & 7 \\ 0 & 10 & -5 & 35 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_2]{5R_2-R_3\to R_3} \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1+2R_2\to R_1} \begin{bmatrix} 1 & 0 & 4 & 10 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x+4z=10$

$y-\frac{1}{2}z=\frac{7}{2}$

Let z be t,

$x=10-4t$

$y=\frac{7}{2}+\frac{1}{2}t$

$z=t$

43. $\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$

$$\left[\begin{array}{cccc} 1 & -1 & 3 & 3 \\ 4 & -8 & 32 & 24 \\ 2 & -3 & 11 & 4 \end{array}\right]$$

$$\Big\downarrow \; \frac{R_2 - 4R_1 \to R_2}{R_3 - 2R_1 \to R_3}$$

$$\left[\begin{array}{cccc} 1 & -1 & 3 & 3 \\ 0 & -4 & 20 & 12 \\ 0 & -1 & 5 & -2 \end{array}\right]$$

$$\Big\downarrow \; -\frac{1}{4}R_2$$

$$\left[\begin{array}{cccc} 1 & -1 & 3 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & -1 & 5 & -2 \end{array}\right]$$

$$\Big\downarrow \; R_3 + R_2 \to R_3$$

$$\left[\begin{array}{cccc} 1 & -1 & 3 & 3 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$\therefore$ Inconsistent linear system

49.

$$4x - 3y + z = -8$$
$$-2x + y - 3z = -4$$
$$x - y + 2z = 3$$

$$\begin{bmatrix} 4 & -3 & 1 & -8 \\ -2 & 1 & -3 & -4 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

$$\Big\downarrow \tfrac{1}{4}R_1$$

$$\begin{bmatrix} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ -2 & 1 & -3 & -4 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

$$\Big\downarrow \frac{R_2 + 2R_1 \to R_2}{R_3 - R_1 \to R_3}$$

$$\begin{bmatrix} 1 & -\frac{3}{4} & \frac{1}{4} & -2 \\ 0 & -\frac{1}{2} & -\frac{5}{2} & -8 \\ 0 & -\frac{1}{4} & \frac{7}{4} & 5 \end{bmatrix}$$

$$\Big\downarrow \frac{-2R_2}{-4R_3}$$

$$\begin{bmatrix} 1 & -3/4 & 1/4 & -2 \\ 0 & 1 & 5 & 16 \\ 0 & 1 & -7 & -20 \end{bmatrix}$$

$$\Big\downarrow -\tfrac{1}{7}R_3$$

$$\begin{bmatrix} 1 & -3/4 & 1/4 & -2 \\ 0 & 1 & 5 & 16 \\ 0 & -\frac{1}{7} & 1 & \frac{20}{7} \end{bmatrix}$$

$$\Big\downarrow \tfrac{1}{7}R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & -3/4 & 1/4 & -2 \\ 0 & 1 & 5 & 16 \\ 0 & 0 & \frac{12}{7} & \frac{36}{7} \end{bmatrix}$$

$$\Big\downarrow \tfrac{7}{12}R_3$$

$$\begin{bmatrix} 1 & -3/4 & 1/4 & -2 \\ 0 & 1 & 5 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow \frac{R_2 - 5R_3 \to R_2}{R_1 - \frac{1}{4}R_3 \to R_1}$$

$$\begin{bmatrix} 1 & -3/4 & 0 & -11/4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow R_1 + \tfrac{3}{4}R_2 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore x = -2$, $y = 1$, $z = 3$

11.1

Form of a Matrix

Elementary Row Operations

Back Substitution

Linear Systems with One Solution

Dependent or Inconsistent Linear Systems

Solving a Linear System

Applications

1. Linear Systems 29-48
2. Solving a Linear System

14. $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

(a) Yes

(b) No

(c) $x + 3y = -3$

$y = 5$

2

21. $\begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 1 & 1 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$

$\downarrow 3R_1 + R_2 \to R_2$

$\begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 4 & 7 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$

3 26. $\begin{bmatrix} 1 & 1 & -3 & 8 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

(a) $x + y - 3z = 8$

$y - 3z = 5$

$z = -1$

(b) $y - 3z = 5$

$y - 3(-1) = 5$

$y = 2$

$x + y - 3z = 8$

$x + (2) - 3(-1) = 8$

$x + 5 = 8$

$x = 3$

④ 30. $\begin{cases} x+y+6z=3 \\ x+y+3z=3 \\ x+2y+4z=7 \end{cases}$

$$\left[\begin{array}{cccc} 1 & 1 & 6 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 2 & 4 & 7 \end{array}\right]$$

Gaussian Elimination

$$\Big\downarrow \frac{R_2 - R_1 \to R_2}{R_3 - R_1 \to R_3}$$

$$\left[\begin{array}{cccc} 1 & 1 & 6 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_1 \to R_2 \\ R_2 - R_1 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 1 & 6 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\Big\downarrow -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{cccc} 1 & 1 & 6 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$z=0$

$y-2z=4$

$y-0=4$

$y=4$

$x+y+6z=3$

$x+4+6(0)=3$

$x=-1$

$\therefore (-1, 4, 0)$

29. $\begin{cases} x-2y+z=1 \\ y+2z=5 \\ x+y+3z=8 \end{cases}$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 8 \end{bmatrix}$$

$$\Big\downarrow R_3-R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 2 & 7 \end{bmatrix}$$

$$\Big\downarrow R_3-3R_2 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -4 & -8 \end{bmatrix}$$

$$\Big\downarrow -\tfrac{1}{4}R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$z=2$

$y+2z=5$

$y+2(2)=5$

$y=1$

$x-2y+z=1$

$x-2(1)+(2)=1$

$x=1$

$\therefore (1,1,2)$

31. $\begin{cases} x+y+z=2 \\ 2x-3y+2z=4 \\ 4x+y-3z=1 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -3 & 2 & 4 \\ 4 & 1 & -3 & 1 \end{bmatrix}$$

$$\Big\downarrow \; 2R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 5 & 0 & 0 \\ 4 & 1 & -3 & 1 \end{bmatrix}$$

$$\Big\downarrow \; \dfrac{4R_1 - R_3 \to R_3}{\frac{1}{5}R_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 7 & 7 \end{bmatrix}$$

$$\Big\downarrow \; R_3 - 3R_2 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix}$$

$$\Big\downarrow \; \tfrac{1}{7}R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Big\downarrow \; \dfrac{R_1 - R_2 \to R_1}{R_1 - R_3 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore x=1,\ y=0,\ z=1$

32. $\begin{cases} x+y+z=4 \\ -x+2y+3z=17 \\ 2x-y=-7 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ -1 & 2 & 3 & 17 \\ 2 & -1 & 0 & -7 \end{bmatrix} \xrightarrow[2R_1-R_3\to R_3]{R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & 4 & 21 \\ 0 & 3 & 2 & 15 \end{bmatrix}$$

$$\xrightarrow{R_2-R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & 4 & 21 \\ 0 & 0 & 2 & 6 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{3}R_2} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 4/3 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow[R_1-R_3\to R_1]{R_2-\frac{4}{3}R_3\to R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1-R_2\to R_1} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore x=-2,\ y=3,\ z=3$

33. $\begin{cases} x + 2y - z = -2 \\ x + z = 0 \\ 2x - y - z = -3 \end{cases}$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & -3 \end{array}\right]$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 2 & -1 & -1 & -3 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{2}$

$$\downarrow \begin{array}{c} R_3 - R_1 \to R_3 \\ \hline R_2 - 2R_1 \to R_2 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & -5 & 1 & 1 \\ 0 & -2 & 2 & 2 \end{array}\right]$$

$\boxed{2}$

$-\frac{1}{5}R_2 \quad \downarrow R_3 - \frac{2}{5}R_2 \to R_3 \quad -\frac{1}{2}R_3$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1/5 & -1/5 \\ 0 & 0 & 8/5 & 8/5 \end{array}\right]$$

$$\downarrow \frac{5}{8}R_3$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & -2 \\ 0 & 1 & -1/5 & -1/5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$z = 1$

$y - \frac{1}{5}z = -\frac{1}{5}$

$y = -\frac{1}{5} + \frac{1}{5} = 0$

$x + 2y - z = -2$

$x + 0 - 1 = -2$

$x = -1$

$\therefore (-1, 0, 1)$

34. $\begin{cases} 2y + z = 4 \\ x + y = 4 \\ 3x + 3y - z = 10 \end{cases}$

$$\begin{bmatrix} 0 & 2 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 3 & 3 & -1 & 10 \end{bmatrix}$$

$\downarrow R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 2 & 1 & 4 \\ 3 & 3 & -1 & 10 \end{bmatrix}$$

$\downarrow R_3 - 3R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$\downarrow \dfrac{\frac{1}{2}R_2}{-R_3}$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$z = 2$

$y + \frac{1}{2}z = 2$

$y + \frac{1}{2}(2) = 2$

$y = 1$

$x + y = 4$

$x = 4 - 1$

$= 3$

$\therefore (3, 1, 2)$

(5) Dependent or Inconsistent Linear Systems

40. $\begin{cases} x + 3z = 3 \\ 2x + y - 2z = 5 \\ -y + 8z = 8 \end{cases}$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 2 & 1 & -2 & 5 \\ 0 & -1 & 8 & 8 \end{bmatrix}$$

$$\Big\downarrow R_2 - 2R_1 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -8 & -1 \\ 0 & -1 & 8 & 8 \end{bmatrix}$$

$$\Big\downarrow R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -8 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$\therefore$ No solution, Inconsistent

⑥ Solving a Linear System

50. $$\begin{cases} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{cases}$$

$$\begin{bmatrix} 2 & -3 & 5 & 14 \\ 4 & -1 & -2 & -17 \\ -1 & -1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow[R_3 + \frac{1}{2}R_1 \to R_3]{R_2 - 2R_1 \to R_2}$$

$$\begin{bmatrix} 2 & -3 & 5 & 14 \\ 0 & 5 & -12 & -45 \\ 0 & -5/2 & 7/2 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3 + \frac{1}{2}R_2 \to R_3]{\frac{1}{2}R_1}$$

$$\begin{bmatrix} 1 & -3/2 & 5/2 & 7 \\ 0 & 5 & -12 & -45 \\ 0 & 0 & -5/2 & -\frac{25}{2} \end{bmatrix}$$

$$\xrightarrow[-\frac{2}{5}R_3]{\frac{1}{5}R_2}$$

$$\begin{bmatrix} 1 & -3/2 & 5/2 & 7 \\ 0 & 1 & -12/5 & -9 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$z = 5$

$y - \frac{12}{5}(5) = 9$

$y = -9 + 12$
$= 3$

$x - \frac{3}{2}(3) + \frac{5}{2}(5) = 7$

$x - \frac{9}{2} + \frac{25}{2} = 7$

$x + 8 = 7$

$x = -1$

$$2x - 3y + 5z = 14$$

$$2\left(-\frac{1}{5}\right) - 3\left(\frac{1}{5}\right) + 5(3)$$

$$= -\frac{2}{5} - \frac{3}{5} + \frac{75}{5}$$

$$= \frac{70}{5}$$

$$= 14$$

$$4x - y - 2z = -17$$

$$4\left(-\frac{1}{5}\right) - \left(\frac{1}{5}\right) + 2(3)$$

$$= -\frac{4}{5} - \frac{1}{5} + \frac{30}{5}$$

$$= \frac{25}{5}$$

$$= 5$$

69.

$$5x + 10y + 15z = 50$$

$$15x + 20y + 0 = 50$$

$$10x + 10y + 10z = 50$$

$$\begin{bmatrix} 5 & 10 & 15 & 50 \\ 15 & 20 & 0 & 50 \\ 10 & 10 & 10 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 10 \\ 3 & 4 & 0 & 10 \\ 2 & 2 & 2 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & -2 & -9 & -20 \\ 0 & -2 & -4 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3}$$

$$\begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & -2 & -9 & -20 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{5}R_3]{-\frac{1}{2}R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & 1 & \frac{9}{2} & 10 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$z = 2$ $\quad y = 10 - \frac{9}{2}(2)$ $\quad x = 10 - 3(2) - 2(1)$

$= 1$ $\quad = 2$

$\therefore (2, 1, 2)$

11.2 The Algebra of Matrices

① Equality of Matrices

② Matrix Operations

③ Matrix Equations

④ Linear Systems as Matrix Equations

⑤ Applications

②

2

Matrix Operations

8/3/2024

9. $$\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$$

10. $$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

11. $$3\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & -3 \\ 3 & 0 \end{bmatrix}$$

12. $$2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$\therefore$ not possible, because you cannot add a 3x3 matrix and a 3x2 matrix together

13. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 6 \\ -2 & 0 \end{bmatrix}$

$\therefore$ not possible, 3×2 matrix and a 3×2 matrix are not allowed to be multiplied together

14. $\begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2(1)+1(3)+2(-2) & 2(-2)+1(6)+2(0) \\ 6(1)+3(3)+4(-2) & 6(-2)+3(6)+4(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 7 & 6 \end{bmatrix}$$

15. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & -2+4 & 3-2 \\ -1+8 & 2+8 & -3-4 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 2 & 1 \\ 7 & 10 & -7 \end{bmatrix}$$

16. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 - 3 \\ 0 + 1 \\ 5 + 2 \end{bmatrix}$

$= \begin{bmatrix} 7 \\ 1 \\ 7 \end{bmatrix}$

32. (a) $B^2 = \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix}$

$\therefore$ 2×3 matrix and 2×3 matrix incompatible

(b) $F^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

33. (a) $A^2 = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 16+6 & 24+18 \\ 4+3 & 6+9 \end{bmatrix}$

$= \begin{bmatrix} 22 & 42 \\ 7 & 15 \end{bmatrix}$

(b) $A^3 = A^2 A$

$$= \begin{bmatrix} 22 & 42 \\ 7 & 15 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 130 & 132+126 \\ 43 & 87 \end{bmatrix}$$

$$= \begin{bmatrix} 130 & 258 \\ 43 & 87 \end{bmatrix}$$

34. (a) $(DA)B$

$$= \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -14 \end{bmatrix} \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 21 & 28 \end{bmatrix}$$

(b) $D(AB)$ $\quad AB = \begin{bmatrix} 1 & 6 & -5 \\ 7 & -7 & 21 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 & -5 \\ 7 & -7 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 21 & 28 \end{bmatrix}$$

③ Matrix Equations

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

17. $2X + A = B$

$$2X = B - A$$

$$= \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 2 \end{bmatrix}$$

18. $3X - B = C$

$3X = C + B$

$\therefore$ no solution, C and B cannot be added together as their dimensions are different

19. $2(B - X) = D$

$$2\left(\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - X\right) = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

$$2\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - 2X = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

$\therefore$ no solution because B is 2×2, and D is 3×2

20. $5(X - C) = D$

$5X - 5C = D$

$$5X - 5\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

$$5X = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix} + \begin{bmatrix} 10 & 15 \\ 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$5X = \begin{bmatrix} 20 & 35 \\ 35 & 20 \\ 10 & 10 \end{bmatrix}$$

$$X = \frac{1}{5} \begin{bmatrix} 20 & 35 \\ 35 & 20 \\ 10 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 7 \\ 7 & 4 \\ 2 & 2 \end{bmatrix}$$

④ Linear Systems as Matrix Equations

47. $\begin{cases} 2x - 5y = 7 \\ 3x + 2y = 4 \end{cases}$

$$\begin{bmatrix} 2 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

48. $\begin{cases} 6x - y + z = 12 \\ 2x \quad + z = 7 \\ \quad y - 2z = 4 \end{cases}$

$$\begin{bmatrix} 6 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \\ 4 \end{bmatrix}$$

49. $\begin{cases} 3x_1 + 2x_2 - x_3 + x_4 = 0 \\ x_1 \quad - x_3 \quad = 5 \\ \quad 3x_2 + x_3 - x_4 = 4 \end{cases}$

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

5 Applications

5. $A = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 6 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 6 \end{bmatrix}$

$\boxed{2}$ $\because$ matrices have different dimensions, so they cannot be equal

7. $A = \begin{bmatrix} 3 & 4 \\ -1 & a \end{bmatrix}$ $B = \begin{bmatrix} b & 4 \\ -1 & -5 \end{bmatrix}$

$a = -5$, $b = 3$

Matrix Operations

9. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2+(-1) & 6+(-3) \\ -5+6 & 3+2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$

Matrix Equations

$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

17. $2X + A = B$

$2X = B - A$

$X = \frac{1}{2}(B - A)$

$= \frac{1}{2}\begin{bmatrix} 2-4 & 5-6 \\ 3-1 & 7-3 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} -2 & -1 \\ 2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 2 \end{bmatrix}$$

Matrix Operations

23. (a) $B + C$

$$= \begin{bmatrix} 3 & 1/2 & 5 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -5/2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 5 \\ 1 & 1 & 0 \end{bmatrix}$$

(b) $B + F$

$\therefore$ cannot be performed because the matrices have different dimensions,

B is 2 by 3 while F is 3 by 3

27. (a) AD

$$AD = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 7 & 3 \end{bmatrix}$$

$\therefore$ undefined,

2x2 matrix and 1x2 matrix

(b) DA

$$DA = \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 14 + 0 \\ -35 + 21 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ -14 \end{bmatrix}$$

Equality of Matrices

43. $\begin{bmatrix} x & 2y \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2x & -6y \end{bmatrix}$

$x = 2$ $\quad 2y = -2$

$4 = 2x$ $\quad 6 = -6y$

$x = 2$ $\quad y = -1$

Linear Systems as Matrix Equations

47. $2x - 5y = 7$

$3x + 2y = 4$

$$\begin{bmatrix} 2 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Applications

53.

$$\begin{bmatrix} 0.75 & 0.10 & 0 \\ 0.25 & 0.70 & 0.70 \\ 0 & 0.20 & 0.30 \end{bmatrix} = A$$

$$\begin{bmatrix} 4 \\ 20 \\ 10 \end{bmatrix} = B$$

(a)

$$AB = \begin{bmatrix} 0.75 & 0.10 & 0 \\ 0.25 & 0.70 & 0.70 \\ 0 & 0.20 & 0.30 \end{bmatrix} \begin{bmatrix} 4 \\ 20 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 \cdot 4 + 0.10 \cdot 20 + 0 \\ 1 + 14 + 7 \\ 0 + 4 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 2 + 0 \\ 1 + 14 + 7 \\ 0 + 4 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 22 \\ 7 \end{bmatrix}$$

(b) Matrix AB shows the number of members in each category of years of postsecondary education.

5 no education, and so on

11.3 Inverses of Matrices and Matrix Equations

① Verifying the Inverse of a Matrix

② The Inverse of a 2X2 Matrix

③ Finding the Inverse of a Matrix

④ Solving a Linear System

⑤ Skills Plus

③ Finding the Inverse of a Matrix

① Verifying the Inverse of a Matrix

3. $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8-7 & -4+4 \\ 14-14 & -7+8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-7 & 2-2 \\ -28+28 & -7+8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$ $B = \begin{bmatrix} 7/2 & -3/2 \\ 2 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 7/2 & -3/2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-6 & -3+3 \\ 14-14 & -6+7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 7/2 & -3/2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 7-6 & -\frac{21}{2}+\frac{21}{2} \\ 4-4 & -6+7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

② The Inverse of a 2×2 Matrix

7. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{14-12}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 7-6 & -14+14 \\ 3-3 & -6+7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + 2R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 5 & 4 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & 5 & 4 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 5R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & -1 & 2 & -5/2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -2 & 5/2 & -1 \end{array}\right]$$

$$\xrightarrow[R_1 - 2R_3 \to R_1]{R_2 - R_3 \to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 5 & -5 & 2 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -2 & 5/2 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2 \to R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -2 & 5/2 & -1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -2 & 1 \\ -2 & 5/2 & -1 \end{bmatrix}$$

$$BB^{-1} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 2 & -2 & 1 \\ -2 & 5/2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Finding the Inverse of a Matrix

11. $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-9+10}\begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$$

12. $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} 9 & -4 \\ -7 & 3 \end{bmatrix}$$

$$= \frac{1}{27-28}\begin{bmatrix} 9 & -4 \\ -7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}$$

13. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-26+25}\begin{bmatrix} -13 & -5 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix}$$

14. $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$

$$A^{-1} = \frac{1}{35-32}\begin{bmatrix} -5 & -4 \\ -8 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{3} & -4/3 \\ -8/3 & -7/3 \end{bmatrix}$$

15. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{24-24}$$

$\therefore A^{-1}$ does not exist

16. $\begin{bmatrix} 1/2 & 1/3 \\ 5 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{2-\frac{5}{3}}\begin{bmatrix} 4 & -1/3 \\ -5 & 1/2 \end{bmatrix}$$

$$= 3\begin{bmatrix} 4 & -1/3 \\ -5 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -1 \\ -15 & 3/2 \end{bmatrix}$$

17. $\begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{0.24+0.36}\begin{bmatrix} 0.6 & 1.2 \\ -0.3 & 0.4 \end{bmatrix}$$

$$= \frac{1}{0.60}\begin{bmatrix} 0.6 & 1.2 \\ -0.3 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1/2 & 2/3 \end{bmatrix}$$

18. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = M_{31}A_{31} + M_{32}A_{32} + M_{33}A_{33}$$

$$= (-1)^4 \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}(1) + 0$$

$$+ (-1)^6 \begin{vmatrix} 4 & 2 \\ 3 & 3 \end{vmatrix}(1)$$

$$= -5 + 6$$

$$= 1$$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 3 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \frac{R_2 - 3R_3 \to R_2}{4R_3 - R_1 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & -3 \\ 0 & -2 & 1 & -1 & 0 & 4 \end{array}\right]$$

$$\xrightarrow[3R_3+2R_2 \to R_3]{\frac{1}{4}R_1,\ \frac{1}{3}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/4 & 1/4 & 0 & 0 \\ 0 & 1 & -1/3 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 6 \end{array}\right]$$

$$\xrightarrow[R_1 - \frac{3}{4}R_3 \to R_1]{R_2 + \frac{1}{3}R_3 \to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 5/2 & -3/2 & -9/2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -3 & 2 & 6 \end{array}\right]$$

$$\xrightarrow{R_1 - \frac{1}{2}R_2 \to R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -3 & 2 & 6 \end{array}\right]$$

④ Solving a Linear system

39. $-3x-5y=4$

$2x+3y=0$

$$\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-9+16} \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

40. $3x+4y=10$

$7x+9y=20$

$$\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{27-28} \begin{bmatrix} 9 & -4 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -10 \\ 10 \end{bmatrix}$$

47. $\begin{cases} x + y - 2z = 3 \\ 2x + 5z = 11 \\ 2x + 3y = 12 \end{cases}$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 2 & 0 & 5 & 11 \\ 2 & 3 & 0 & 12 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 2 & 3 & 0 & 12 \\ 2 & 0 & 5 & 11 \end{bmatrix}$$

$$\Big\downarrow R_2 - R_3 \to R_3$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 2 & 3 & 0 & 12 \\ 0 & 3 & -5 & 1 \end{bmatrix}$$

$$\Big\downarrow R_2 - 2R_1 \to R_2$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 3 & -5 & 1 \end{bmatrix}$$

$$\Big\downarrow R_3 - 3R_2 \to R_3$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -17 & -17 \end{bmatrix}$$

$$\Big\downarrow -\tfrac{1}{17}R_3$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$z = 1$

$y + 4z = 6$

$y = 6 - 4$

$= 2$

$x + y - 2z = 3$

$x = 3 - 2 + 2$

$= 3$

$\therefore (3, 2, 1)$

11.3 Inverses of Matrices and Matrix Equations

1. (a) identity

(b) A, A

(c) Inverse matrix

3. $A=\begin{bmatrix}4 & 1\\ 7 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\ -7 & 4\end{bmatrix}$

$$AB=\begin{bmatrix}4 & 1\\ 7 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ -7 & 4\end{bmatrix}$$

$$=\begin{bmatrix}8-7 & -4+4\\ 14-14 & -7+8\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$BA=\begin{bmatrix}2 & -1\\ -7 & 4\end{bmatrix}\begin{bmatrix}4 & 1\\ 7 & 2\end{bmatrix}$$

$$=\begin{bmatrix}8-7 & 2-2\\ -28+28 & -7+8\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$\because AB = BA = I_2$

$\therefore B$ is the inverse of A

7. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{14-12}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3/2 & 7/2 \end{bmatrix}$$

$$= \begin{bmatrix} 7-6 & -14+14 \\ 3-3 & -6+7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & -2 \\ -3/2 & 7/2 \end{bmatrix}\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7-6 & 4-4 \\ -\frac{21}{2}+\frac{21}{2} & -6+7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore AA^{-1} = A^{-1}A = I_2$$

19. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1/2 & \frac{1}{2} & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+R_3\to R_3]{R_1+R_2\to R_2}\left[\begin{array}{ccc|ccc} 1 & 2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3 & -1/2 & 1/2 & 1 & 0 \\ 0 & 5 & -1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2}\left[\begin{array}{ccc|ccc} 1 & 2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1/6 & 1/6 & 1/3 & 0 \\ 0 & 5 & -1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-5R_2\to R_3}\left[\begin{array}{ccc|ccc} 1 & 2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1/6 & 1/6 & 1/3 & 0 \\ 0 & 0 & -1/6 & -5/6 & -2/3 & 1 \end{array}\right]$$

$$\xrightarrow{-6R_3}\left[\begin{array}{ccc|ccc} 1 & 2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1/6 & 1/6 & 1/3 & 0 \\ 0 & 0 & 1 & 5 & 4 & -6 \end{array}\right]$$

$$\xrightarrow[R_2+\frac{1}{6}R_3\to R_2]{R_1-\frac{1}{2}R_3\to R_1}\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & -2 & 3 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 5 & 4 & -6 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2\to R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -4 & 5 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 5 & 4 & -6 \end{array}\right]$$

$\therefore$ Inverse: $\begin{bmatrix} -4 & -4 & 5 \\ 1 & 1 & -1 \\ 5 & 4 & -6 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & -1 & 0 & 1 & 0 \\ 1 & -1 & -10 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - 4R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -13 & -4 & 1 & 0 \\ 0 & -3 & -13 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -13 & -4 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 1 \end{array}\right]$$

$\therefore$ Last row all 0's, so the matrix does not have an inverse

Solving a Linear System as a Matrix Equation

39. $\begin{cases} -3x - 5y = 4 \\ 2x + 3y = 0 \end{cases}$

Let $A = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \end{bmatrix}$,

$B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$$AX = B$$

$$\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}^{-1}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-9 + 10} \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 0 \\ -8 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$\therefore x = 12$, $y = -8$

61. (a) $\begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix} = A$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 4 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - \frac{4}{3}R_1 \to R_2} \left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2/3 & 0 & -4/3 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{3}R_1]{\frac{3}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1/3 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -2 & 3/2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1/3 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -2 & 3/2 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 1/3 & 0 & -2/3 & 3/2 & -1 \\ 0 & 1 & 0 & -2 & 3/2 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - \frac{1}{3}R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 3/2 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 1 \end{array}\right]$$

(b) $B = \begin{bmatrix} 10 \\ 14 \\ 13 \end{bmatrix}$

$$AX = B$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 13 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3/2 & 0 \\ 1 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 14 - 13 \\ -20 + 21 + 0 \\ 10 - 21 + 13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore x = 1,\ y = 1,\ z = 2$

(c) $$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3/2 & 0 \\ 1 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 12 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 12 - 10 \\ -18 + 18 + 0 \\ 9 - 18 + 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

(d) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3/2 & 0 \\ 1 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 4 - 11 \\ -4 + 6 + 0 \\ 2 - 6 + 11 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 2 \\ 7 \end{bmatrix}$$

$\therefore$ No, because $x = -7$.

11.4 Determinants and Cramer's Rule

① Finding Determinants

② Minors and Cofactors

③ Cramer's Rule

④ Skills Plus

① Finding Determinants

② Minors and Cofactors

① Finding Determinants

5. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = ad - bc$$
$$= 6 - 0$$
$$= 6$$

6. $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

$$\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2$$
$$= 2$$

7. $\begin{bmatrix} \frac{3}{2} & 1 \\ -1 & -\frac{2}{3} \end{bmatrix}$

$$\begin{vmatrix} \frac{3}{2} & 1 \\ -1 & -\frac{2}{3} \end{vmatrix} = ad - bc$$
$$= -1 + 1$$
$$= 0$$

8. $\begin{bmatrix} 0.2 & 0.4 \\ -0.4 & -0.8 \end{bmatrix}$

$$\begin{vmatrix} 0.2 & 0.4 \\ -0.4 & -0.8 \end{vmatrix}$$

$$= 0.2(-0.8) - 0.4(-0.4)$$

$$= -0.16 + 0.16$$

$$= 0$$

9. $\begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$

$$\begin{vmatrix} 4 & 5 \\ 0 & -1 \end{vmatrix} = ad - bc = -4 - 0$$

$$= -4$$

10. $\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} = -2 \cdot -2 - 1 \cdot 3$

$$= 1$$

21. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 2\cdot(-1)^2 \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} + 1\cdot(-1)^3 \begin{vmatrix} 0 & 4 \\ 0 & -3 \end{vmatrix}$$

$$+ 0\cdot(-1)^4 \begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix}$$

$$= 2(2) + 0 + 0$$

$$= 4$$

25. $$\begin{vmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{vmatrix} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

$$= 0 + 2(-1)^5 \begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix}$$

$$+ 2(-1)^6 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= 0 - 2(-6) + 2(0 - 6)$$

$$= 12 - 12$$

$$= 0 \quad \therefore \text{ no inverse}$$

② Minors and Cofactors

$$A = \begin{bmatrix} 1 & 0 & 1/2 \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

15. $M_{11} = 20 - 0$
$= 20$

$A_{11} = (-1)^2 M_{11}$
$= 20$

16. $M_{33} = 5 - 0$
$= 5$

$A_{33} = (-1)^6 M_{33}$
$= 5$

17. $M_{12} = -3(4) - 2(0)$
$= -12$

$A_{12} = (-1)^3 M_{12}$
$= -1(-12)$
$= 12$

18. $M_{13} = \begin{vmatrix} -3 & 5 \\ 0 & 0 \end{vmatrix} = 0$

$A_{13} = (-1)^4 M_{13} = 0$

19. $M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$A_{23} = (-1)^5 M_{23} = 0$

20. $M_{32} = \begin{vmatrix} 1 & 1/2 \\ -3 & 2 \end{vmatrix} = 2 + \frac{3}{2} = \frac{7}{2}$

$A_{32} = (-1)^5 M_{32} = -\frac{7}{2}$

③ Cramer's Rule

41. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

$$D = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$D_x = \begin{bmatrix} -9 & -1 \\ 8 & 2 \end{bmatrix}$$

$$D_y = \begin{bmatrix} 2 & -9 \\ 1 & 8 \end{bmatrix}$$

$$x = \frac{|D_x|}{|D|} = \frac{-18+8}{4+1} = \frac{-10}{5} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{16+9}{4+1} = \frac{25}{5} = 5$$

$\therefore x = -2, 5$

42. $\begin{cases} 6x+12y=33 \\ 4x+7y=20 \end{cases}$

$D=\begin{bmatrix} 6 & 12 \\ 4 & 7 \end{bmatrix}$ $D_x=\begin{bmatrix} 33 & 12 \\ 20 & 7 \end{bmatrix}$ $D_y=\begin{bmatrix} 6 & 33 \\ 4 & 20 \end{bmatrix}$

$x=\dfrac{|D_x|}{|D|}$ $y=\dfrac{|D_y|}{|D|}$

$=\dfrac{231-240}{42-48}$ $=\dfrac{120-132}{42-48}$

$=\dfrac{3}{2}$ $=\dfrac{-12}{-6}$

$=2$

43. $\begin{cases} x-6y=3 \\ 3x+2y=1 \end{cases}$

$D=\begin{bmatrix} 1 & -6 \\ 3 & 2 \end{bmatrix}$ $Dx=\begin{bmatrix} 3 & -6 \\ 1 & 2 \end{bmatrix}$ $D_y=\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

$x=\dfrac{|D_x|}{|D|}$ $y=\dfrac{|D_y|}{|D|}$

$=\dfrac{1-9}{20}$

$=\dfrac{6+6}{2+18}$ $=-\dfrac{2}{5}$

$=\dfrac{3}{5}$

5. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 6 - 0$$

$$= 6$$

21. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 2\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ 0 & -3 \end{vmatrix} + 0\begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix}$$

$$= 2(6-4)$$

$$= 4$$

$\therefore$ has an inverse as $\begin{vmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{vmatrix} \neq 0$

25. $$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{vmatrix} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

$$= 0 + 2(-1)^5 \begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} + 2(-1)^6 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= -2(8-14) + 2(0-6)$$

$$= -2(-6) + 2(-6)$$

$$= 12 - 12$$

$$= 0$$

35. $$\begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 3 & 0 & 1 & 7 \end{vmatrix}$$

Row 3 − Row 2 → Row 3

$$= \begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 7 \end{vmatrix}$$

$$= 0 + 0 + a_{33} A_{33} + 0$$

$$= 1 (-1)^6 \begin{vmatrix} 0 & 0 & 6 \\ 2 & 1 & 3 \\ 3 & 0 & 7 \end{vmatrix}$$

$$= 0 + 0 + a_{13} (-1)^4 \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 6 (0 - 3)$$

$$= -18$$

41. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

$$|D| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 4 - (-1)$$

$$= 5$$

$$|D_x| = \begin{vmatrix} -9 & -1 \\ 8 & 2 \end{vmatrix}$$

$$= -18 - (-8)$$

$$= -10$$

$$|D_y| = \begin{vmatrix} 2 & -9 \\ 1 & 8 \end{vmatrix}$$

$$= 16 - (-9 \cdot 1)$$

$$= 25$$

$$x = \frac{|D_x|}{|D|} = \frac{-10}{5} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{25}{5} = 5$$

47. $\begin{cases} x - y + 2z = 0 \\ 3x + z = 11 \\ -x + 2y = 0 \end{cases}$

$$|D| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$= 3(-1)^3 \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} + 0 + 1(-1)^5 \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= -3(0-4) - 1(2-1)$$

$$= 12 - 1 = 11$$

$$|D_x| = \begin{vmatrix} 0 & -1 & 2 \\ 11 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= a_{32} A_{32}$$

$$= 2 \cdot (-1)^5 \begin{vmatrix} 0 & 2 \\ 11 & 1 \end{vmatrix}$$

$$= -2(0-22)$$

$$= 44$$

$$|D_y| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 11 & 1 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= a_{31} A_{31}$$

$$= -1 (-1)^4 \begin{vmatrix} 0 & 2 \\ 11 & 1 \end{vmatrix}$$

$$= -1(0-22)$$

$$= 22$$

$$|D_z| = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 0 & 11 \\ -1 & 2 & 0 \end{vmatrix}$$

Row 1 + Row 3 → Row 1

$$= \begin{vmatrix} 0 & 1 & 0 \\ 3 & 0 & 11 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= a_{12} A_{12}$$

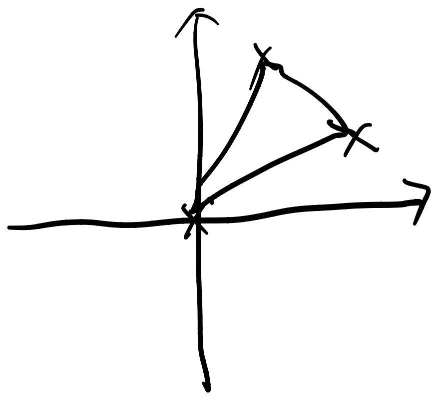
$$= 1 (-1)^3 \begin{vmatrix} -3 & 11 \\ -1 & 0 \end{vmatrix}$$

$$= -1(0+11)$$

$$= -11$$

$$x = \frac{|D_x|}{|D|} = \frac{44}{11} = 4 \qquad y = \frac{|D_y|}{|D|} = \frac{22}{11} = 2 \qquad z = \frac{|D_z|}{|D|} = \frac{-11}{11} = -1$$

57.



$$\therefore A = \pm\frac{1}{2}\left|\begin{matrix} 0 & 0 & 1 \\ 6 & 2 & 1 \\ 3 & 8 & 1 \end{matrix}\right|$$

$$= \pm\frac{1}{2}\left(a_{13}A_{13}\right)$$

$$= \pm\frac{1}{2}\left(1\cdot(-1)^4\left|\begin{matrix} 6 & 2 \\ 3 & 8 \end{matrix}\right|\right)$$

$$= \pm\frac{1}{2}(42)$$

$$= 21$$